

A (SHORT) SURVEY ON DOMINATED SPLITTING

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ABSTRACT. We present here the concept of Dominated Splitting and give an account of some important results on its dynamics.

Dedicated to Ricardo Mañé (1948-1995)

1. INTRODUCTION

Since the invention of Calculus a basic tool in order to get information on a smooth object is to look to its “linear approximation”. Let me start with two very simple examples to give a rough idea of the purpose of the present paper.

First, as we learn during the first courses, when we have a smooth function of the real line or the interval, if we know that at some point x , the derivative $f'(x)$ has a definite sign, then we know that the function is strictly monotonous on a neighborhood of the point x . Notice that any map g smoothly close to f will have $g'(x)$ of the same sign. On the other hand if $f'(x) = 0$ then we cannot ensure what the behavior of f around x will be. However, if we know that on a C^1 -neighborhood of f , all maps have the same behavior around a uniform neighborhood of x , then we conclude that the derivative of f has a definite sign at x .

The second example I would like to mention comes from the study of autonomous differential equations. Let $x' = f(x)$ be a differential equation where $f : \Omega \rightarrow \mathbb{R}^n$ is a smooth map on an open region Ω . Suppose that we have an isolated equilibrium point x_0 and let $A = Df_{x_0}$. We know that if all the eigenvalues of A have negative real part then the equilibrium point x_0 is (asymptotically) stable. Notice that if g smoothly close to f , the corresponding differential equation will have an asymptotically stable equilibrium point near x_0 . On the other hand, if all eigenvalues of A have non positive real part (but there is one with zero real part) then we cannot ensure what the behavior around x_0 will be. However, if we know that for any g on a C^1 neighborhood of f , the corresponding differential equation has a unique equilibrium point in a uniform neighborhood of x_0 and it is asymptotically stable, then all the eigenvalues of A have negative real part.

I mentioned the above examples to support the following idea (or principle) in smooth dynamics, which is somehow the leitmotif of this survey: *a “stable” structure of the linear approximation should allow one to describe the dynamics (or at least should impose restrictions on it) and a “stable” dynamical phenomena implies some “stable” structure on the linear approximation.*

This principle has been beautifully accomplished in the so called *hyperbolic theory* started by D.V. Anosov and S. Smale in the sixties and an endless list of contributors (Bowen, Franks, Mañé, Newhouse, Palis, Pugh, Shub just to mention a few). Let us explain it in a very informal way. A smooth dynamical system $f : M \rightarrow M$ is hyperbolic (on a compact invariant set Λ) if the tangent map Df has a hyperbolic structure over the set, meaning that the tangent bundle splits into two subbundles $T_\Lambda = E^s \oplus E^u$ invariant under Df and the action of Df on this subbundles has a uniform behavior (uniformly contracting on E^s and uniformly expanding on E^u). A precise definition is given in the next section. This hyperbolic structure of Df has strong implications on the dynamics of f . Locally, the dynamics splits into two “directions” and in one direction

points get exponentially close by forward iteration and on the other one points get exponentially close by backward iteration. These facts allow to successfully describe the dynamics (see for instance [S], [Sh], [R], [KH]). On the other hand this hyperbolic structure cannot be destroyed by small smooth perturbation and “stability” can be get from here. Furthermore, (structurally) stable dynamical systems are hyperbolic (a remarkable result by Mañé [M1]).

Weaker forms of hyperbolicity appeared in the literature (in the 70’s): Partial Hyperbolicity and Dominated Splitting. These (non hyperbolic) structures on the tangent map Df of a dynamical system $f : M \rightarrow M$ can not be destroyed by small perturbations. The weaker one having this property is Dominated Splitting. Roughly speaking, we say that an invariant set has Dominated Splitting if the tangent bundle over the set splits into two invariant subbundles which are invariant under Df and although we do not know (a priori) that the action of Df on these subbundles has a uniform behavior (contracting or expanding), we know that the action on one *dominates* the action on the other one (see Definition 2.2). This “domination” prevents the structure to be destroyed by small perturbations. This dominated splitting structure may allow us to think that the dynamics of f splits into two directions. Nevertheless, since the action of Df on the invariant subbundles might not have a uniform behavior we can not get information on how the dynamics on these “direction” would be. And so, trying to get dynamical information from the dominated splitting seems hopeless.

Can we describe the dynamics of a set having Dominated Splitting? After a first glance, the question is very naive since any dynamical system $f : M \rightarrow M$ can be embedded into another one having Dominated Splitting (multiplying by other systems having strong contraction/expansion), though in this case the dynamics lives in low dimensional submanifold. We have to take a deeper look to see which are the right questions to ask. We will see through the following pages there are many results in this direction though there is still a long way to a comprehensive understanding of it.

Dominated Splitting appeared in the literature as a *tool* and during the last twenty years there had been a growing interest on it and it fits with many different subjects in dynamics.

It plays a central role in the understanding of global dynamics from a (C^1) generic viewpoint. The Palis’ conjecture ([Pa],[PaT]) motivated much of the work in this area. The state of the art regarding this conjecture is the remarkable result by S. Crovisier and E. Pujals ([CP]). For a global view on generic C^1 dynamics I recommend the survey by C. Bonatti ([B]) and the comprehensive work by Crovisier in [C1]. Robust dynamical phenomena (in the C^1) topology, as robust transitivity for example, are well described in terms of Dominated Splitting as well and there are some surveys that the interested reader could consult ([PS4], [P1], [P2] and also [B] and the book [BDV]).

It plays also a major role in studying C^1 generic diffeomorphisms preserving a volume form that started with the seminal work of R. Mañé ([M6]). As a reference to the subject see the paper by Avila and Bochi [AB]. Recently, Avila-Crovisier-Wilkinson announced a landmark result in this area and the paper will be available soon [ACW]. It is also present in the study of *stable ergodicity* for diffeomorphisms preserving a volume form (see [ACW], [T1], [Wi]).

The extension of the ergodic theory [Bo] of hyperbolic systems (as the existence of SRB or physical measures) to a wider class it has been the interest of researchers for a long time. Systems having Dominated Splitting are in the class where successful extensions has taken place (see for instance [BDV] and references therein).

On the other hand, the (robust) lack of dominated splitting has wild dynamical consequences (see [B], [BD1], [BD2] and [BCS]). For instance, in [BCS], it is proven the Pesin Theory ([Pe] and [BaPe]) dramatically fail in the C^1 topology: there are hyperbolic measures where the stable

and unstable manifolds of any point in the support of the measure are trivial. Nevertheless, much of Pesin theory on hyperbolic measures can be recovered in the C^1 category provided the Oseledec's splitting is a dominated splitting, as claimed in [M6]. For instance in [ABC] it is proven the existence of stable and unstable manifolds and a version of Katok's closing lemma [K] (an early adaptation in two dimensions appeared in [Ga2]); in [G] the Katok's horseshoe construction ([K]) is extended to the C^1 case; and Pesin's entropy formula has been extended as well (see [CCE], [ST] and [T2]).

Likewise, Dominated Splitting has to do with flows, specially in the understanding of the Lorenz attractor and the theory of singular hyperbolic flows (see for instance [MPP] and the book [AP]).

A complete survey on the subject will result in a very long paper or just in a nonsense collection of results. To avoid both I tried to focus on some aspects and results to give the flavor of the subject. What is this survey about? It is about the dynamical consequences we may extract of a set with Dominated Splitting just from the structure itself (without assuming other hypothesis like being far from homoclinic tangencies, heterodimensional cycles, or preserving a measure, etc). And so many important contributions (besides the ones mentioned in the above paragraphs) might not appear here. I apologize for this in advance.

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2. DEFINITIONS AND EXAMPLES

A dynamical systems will be a diffeomorphism $f : M \rightarrow M$ where M is compact riemannian manifold without boundary. We denote by $Diff^r(M)$ the space of C^r diffeomorphisms endowed with the C^r topology. We say that f is C^r generic if belongs to a G_δ set (in the Baire sense).

An ϵ -chain from x to y is a finite sequence $x_0 = x, x_1, \dots, x_n = y$ such that $dist(f(x_i), x_{i+1}) < \epsilon$, for $i = 0, \dots, n-1$. The chain recurrent set $\mathcal{R}(f)$ is the set of points $x \in M$ such that there exists an ϵ -chain from x to itself for any $\epsilon > 0$. A set $\Lambda \subset \mathcal{R}(f)$ is chain-transitive if for any $x, y \in \Lambda$ there is an ϵ -chain from x to y . A chain-recurrent class is a chain-transitive set which is maximal for the inclusion. Chain recurrent classes are compact, invariant and pairwise disjoint. The chain recurrent set is where the dynamic takes place (see [Co]). Other central notions are the non-wandering set $\Omega(f)$, the limit set $L(f)$ and set of periodic points $Per(f)$. It is always true that $\overline{Per(f)} \subset L(f) \subset \Omega(f) \subset \mathcal{R}(f)$ and, from the Connecting Lemma for pseudo-orbits [BC], they are C^1 generically equal to each other.

Definition 2.1. A compact invariant set Λ of a dynamical system $f : M \rightarrow M$ is hyperbolic provided the tangent bundle over Λ splits into two subbundles $T_\Lambda M = E^s \oplus E^u$ such that

- E^s and E^u are invariant by Df , that is $Df(E_x^s) = E_{f(x)}^s$ and $Df(E_x^u) = E_{f(x)}^u$.
- There exists $C > 0$ and $0 < \lambda < 1$ such that for any $x \in \Lambda$

$$\|Df^n_{|E_x^s}\| \leq C\lambda^n, \quad n \geq 0 \quad \text{and} \quad \|Df^{-n}_{|E_x^u}\| \leq C\lambda^n, \quad n \geq 0.$$

A periodic point p is hyperbolic if its orbit $\mathcal{O}(p)$ is a hyperbolic set. The index of p is the dimension of E_p^s . The homoclinic class of a hyperbolic periodic point is the closure of the set of transversal intersection between the stable and unstable manifolds of $\mathcal{O}(p)$.

Let's now recall the meaning of Dominated Splitting. This concept was introduced by Mañé in [M2],[M3]. Apparently this notion already appeared in a letter from Mañé to J. Palis when he was an undergraduate student in Uruguay and before the celebrated conference in Salvador de Bahia in 1971. In this letter Mañé claimed he had a proof of the stability conjecture, something

he would achieve almost twenty years later! Also, Pliss [P11] and Liao [L1] used this notion independently without named it. And in [HPS] this notion is also hidden in the definition of *eventually relatively normal hyperbolicity*.

Definition 2.2. Let Λ be an invariant set of $f : M \rightarrow M$. We say that Λ has Dominated Splitting provided the tangent bundle over Λ splits into two subbundles $T_\Lambda M = E \oplus F$ such that

- (i) E and F are invariant by Df .
- (ii) The subbundles E and F are continuous, i.e., E_x and F_x vary continuously with $x \in \Lambda$.
- (iii) There exist $C > 0$ and $0 < \lambda < 1$ such that for any $x \in \Lambda$

$$(1) \quad \|Df_{/E_x}^n\| \|Df_{F_{f^n(x)}}^{-n}\| \leq C\lambda^n, \quad n \geq 0.$$

A way to understand the above definition is that *any direction not contained in the subbundle E converges exponentially fast to the direction F under iteration of Df* .

Condition (1) is equivalent to the following:

$$(2) \quad \text{There exists } m > 0 \text{ such that } \|Df_{/E_x}^m\| \|Df_{F_{f^m(x)}}^{-m}\| \leq \frac{1}{2}$$

and can be written also in the following form:

$$(3) \quad \text{For any } v_E \in E_x - \{0\} \text{ and } v_F \in F_x - \{0\} : \quad \frac{\|Df_x^m v_E\|}{\|v_E\|} \leq \frac{1}{2} \frac{\|Df_x^m v_F\|}{\|v_F\|}.$$

Also, recalling that the co-norm or mininorm of a linear map A is $m(A) = \|A^{-1}\|^{-1}$ the above can be written as

$$(4) \quad \|Df_{/E_x}^m\| \leq \frac{1}{2} m(Df_{/F_x}^m)$$

The above suggest the terminology: the bundle E *dominates* the bundle F , and sometimes it is written as $E \prec F$. Condition (1) and the equivalent ones also imply that to “see” the domination we might have to wait some iterations.

Notice that the Dominated Splitting does not depends on the riemannian metric, that is, a set having dominated splitting still it has it no matter if we change the metric on the manifold. Nevertheless, the constants C and λ above do depend on the metric. We can always find a riemannian metric on the manifold so that the constant C above is equal to 1, in other words we can “see” the domination in the first step. This is a result by N. Gourmelon [Go1].

Let’s see now an elementary property:

Proposition 2.1. *Assume that Λ has a Dominated Splitting $T_\Lambda M = E \oplus F$. Then this Dominated Splitting can be extended to the closure. Also, condition (ii) is equivalent to the following: the maps $x \rightarrow \dim(E_x)$ and $x \rightarrow \dim(F_x)$ for $x \in \Lambda$ are locally constant. The angle between the subspaces E_x and F_x is bounded away from zero.*

Sketch of proof. Let $x_n \rightarrow x$ be a sequence in Λ such that E_{x_n} and F_{x_n} converges to subspaces \tilde{E}_x and \tilde{F}_x . By defining $\tilde{E}_{f^n(x)} = Df_x(\tilde{E}_x)$ and similarly $\tilde{F}_{f^n(x)}$ we have that \tilde{E} and \tilde{F} satisfy (1). In particular $T_x M = \tilde{E}_x \oplus \tilde{F}_x$. Let y_n be also a sequence in Λ such that $y_n \rightarrow x$ and that E_{y_n} and F_{y_n} converges to subspaces denoted E_x and F_x . Notice that $\dim E_x = \dim \tilde{E}_x$ and $\dim F_x = \dim \tilde{F}_x$. We would like to show that $E_x = \tilde{E}_x$. Otherwise, let $v \in \tilde{E}_x$, $\|v\| = 1$ and

$v = v_E + v_F$ with $v_F \neq 0$. Then

$$\begin{aligned}
\|Df^n_{/\tilde{E}(x)}\| &\geq \|Df^n v\| \geq \|Df^n v_F\| - \|Df^n v_E\| \\
&\geq m(Df^n_{/F(x)})\|v_F\| - \|Df^n_{/E(x)}\|\|v_E\| \\
&= \|Df^n_{/E(x)}\| \left(\frac{m(Df^n_{/F(x)})}{\|Df^n_{/E(x)}\|} \|v_F\| - \|v_E\| \right) \\
&\geq \|Df^n_{/E(x)}\| \left(\frac{\|v_F\|}{C\lambda^n} - \|v_E\| \right)
\end{aligned}$$

and so $\frac{\|Df^n_{/\tilde{E}}\|}{\|Df^n_{/E}\|} \rightarrow \infty$. Interchanging the role of E_x and \tilde{E}_x we get a contradiction. From this one can easily conclude the Proposition. No matter which reasonable definition of angle between subspaces we have, it is obvious that it is bounded away from zero. \square

A set may have many dominated splittings. Let's say that Λ has a dominated splitting $T_\Lambda M = E \oplus F$ of index i if $\dim E_x = i$ for any $x \in \Lambda$. The above Proposition also shows that a dominated splitting of index i on a set Λ is *unique*.

Definition 2.3. Let Λ be an invariant set of $f : M \rightarrow M$. Assume that we have a decomposition $T_\Lambda M = E_1 \oplus E_2 \oplus \dots \oplus E_k$ invariant under Df . We say that it is a Dominated Splitting provided

$$T_\Lambda M = (E_1 \oplus \dots \oplus E_j) \oplus (E_{j+1} \oplus \dots \oplus E_k)$$

is a Dominated Splitting for any $j = 1, \dots, k-1$. When the extremal subbundles E_1 and E_k are uniformly contracting and expanding we call it *partially hyperbolic* (sometimes it is required that just one has a uniform behavior).

Thus, on an invariant set Λ having dominated splitting we can consider the *finest dominated splitting* as the dominated splitting $T_\Lambda M = E_1 \oplus E_2 \oplus \dots \oplus E_k$ such that no E_i can be decomposed again so that the whole decomposition is dominated as well. This notion was introduced in [BDP]. The finest dominated splitting is unique. See Appendix B of [BDV].

Now we may ask: given the finest dominated splitting on a set, can we describe the dynamics? The question is twofold: on one hand what are the dynamical implications of such dominated splitting? and on the other one, which are the dynamical phenomena that prevents having a finer dominated splitting? These are very hard questions and we will try to shed some light on them in the following sections.

Before we see some examples let's characterize the domination in terms of cone fields. For $x \in M$ and $a > 0$, an a -cone of dimension $n - i$ is a subset $\mathcal{C}_a(x)$ of $T_x M$ such that we may find a direct decomposition $T_x M = \tilde{E} \oplus \tilde{F}$ with $\dim \tilde{E} = i, \dim \tilde{F} = n - i$ such that

$$\mathcal{C}_a = \{v \in T_x M : v = v_{\tilde{E}} + v_{\tilde{F}} \text{ such that } \|v_{\tilde{E}}\| \leq a\|v_{\tilde{F}}\|\}.$$

Proposition 2.2. Let Λ be an invariant set of $f : M \rightarrow M$. Then Λ has a Dominated Splitting of index i if and only if there exist a map $a : \Lambda \rightarrow \mathbb{R}^+$ bounded away from zero and infinity, a cone field $\mathcal{C}_{a(x)}(x)$ of dimension $n - i$, a number $0 < \lambda < 1$ and a positive integer n_0 such that

$$Df^{n_0}_x(\mathcal{C}_{a(x)}) \subset \mathcal{C}_{\lambda a(f^{n_0}(x))}.$$

Sketch of Proof. The direct implication follows immediately. For the converse, let's assume for simplicity that $n_0 = 1$. Define

$$E_x = \bigcap_{n \geq 0} Df^{-n}(\mathcal{C}_{a(f^n(x))}) \quad \text{and} \quad F_x = \bigcap_{n \geq 0} Df^n(\mathcal{C}_{a(f^{-n}(x))}).$$

It follows that E_x and F_x are subspaces of dimension i and $n - i$, and invariant under Df . Now consider a tiny cone $\mathcal{C}_\epsilon(x)$ with respect to $T_x M = E_x \oplus F_x$. Then there exists $m > 0$ such that $Df_x^m(\mathcal{C}_{a(x)}) \subset \mathcal{C}_\epsilon(f^m(x))$ and from this one gets the domination. \square

For a more subtle characterization in terms of cones and a spectral gap see [BoG] (see [Wj] as well). A characterization of domination in terms of cones also appeared in [N]. From the above Proposition one easily gets the following important property of dominated splitting: it can be extended to a neighborhood and can not be destroyed by perturbations.

Proposition 2.3. *Let $f \in \text{Diff}^1(M)$ and let Λ be a compact invariant set having Dominated Splitting. Then there exist a compact neighborhood $U(\Lambda)$ of Λ and a neighborhood $\mathcal{U}(f) \subset \text{Diff}^1(M)$ of f such that for any $g \in \mathcal{U}(f)$ the compact set $\bigcap_{n \in \mathbb{Z}} g^n(U)$ has Dominated Splitting.*

Lets see some simple examples to have in mind:

- A hyperbolic set is always a set with Dominated Splitting with $E = E^s$ and $F = E^u$ (although it may have other dominated splittings).
- Let p be a fixed (periodic) point of $f : M \rightarrow M$. Assume Df_p has all eigenvalues of moduli different from $\sigma > 0$ (and having eigenvalues greater and smaller than σ). Then p has a Dominated Splitting.
- Let \mathcal{C} be a smooth (at least C^1) simple closed curve and invariant by $f : M \rightarrow M$ and such that $f|_{\mathcal{C}}$ is conjugated to an irrational rotation. Assume that $T_{\mathcal{C}}M = E^s \oplus T\mathcal{C} \oplus E^u$ invariant under Df where E^s is uniformly contracted and E^u is uniformly expanding. This means that \mathcal{C} is a normally hyperbolic simple closed curve supporting an irrational rotation.
- Mañé Derived from Anosov diffeomorphism on \mathbb{T}^3 [M4]: it is a diffeomorphism $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ such that $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$ is a partially hyperbolic splitting. This diffeomorphism have periodic points of different indices, it is transitive and any C^1 small perturbation is transitive as well. This example is obtained by modifying a linear Anosov map on \mathbb{T}^3 by forcing a fixed point going trough a pitch-fork bifurcation (see also [BV], [PS1], [BFSV] for other properties).
- Bonatti-Viana example on \mathbb{T}^4 [BV]: it is a diffeomorphism $f : \mathbb{T}^4 \rightarrow \mathbb{T}^4$ having a Dominated Splitting $T\mathbb{T}^4 = E^{cs} \oplus E^{cu}$ (which is the finest dominated splitting). It is also obtained by modifying an Anosov on \mathbb{T}^4 with hyperbolic structure $T\mathbb{T}^4 = E^s \oplus E^u$ both bidimensional (see [BV], [BuFi], [T1] for other properties as well).
- Let $f : M \rightarrow M$ where M is two dimensional. Let p be a fixed point having an eigenvalue of modulus less than one and the other is one, $T_p M = E^s \oplus F$. Let $\mathcal{W}^s(p)$ the (strong) stable manifold of p with $T_p \mathcal{W}^s(p) = E^s$. Assume that there is C^1 invariant curve $\mathcal{W}^{cu}(p)$ containing p and $T_p \mathcal{W}^{cu} = F$. Assume that there exists a point x of transversal intersection between $\mathcal{W}^s(p)$ and $\mathcal{W}^{cu}(p)$ and such that $f^{-n}(x) \rightarrow_{n \rightarrow +\infty} p$ (think of a homoclinic class associated to a saddle node fixed point). Then $\Lambda = \{p\} \cup \{f^n(x) : n \in \mathbb{Z}\}$ has a dominated splitting.

3. DOMINATED SPLITTING AND PERIODIC POINTS

In this section we present some general results and ideas on Dominated Splitting (related mostly to periodic points). Periodic points play a fundamental role in dynamics from many different viewpoints. One fundamental problem solved by Pugh [Pu1] and known as Pugh's Closing Lemma states that if x is a nonwandering point of $f : M \rightarrow M$ one can perturb f in the C^1 topology to obtain g so that x is a periodic point for g . However, from the proof one can not conclude that the g -orbit of x shadows the f -orbit of x . This difficulty was solved in [M5].

Theorem 3.1 (Mañé's Ergodic Closing Lemma). *Let $f \in \text{Diff}^1(M)$ and let μ be an invariant probability measure. Then, for any neighborhood $\mathcal{U}(f) \subset \text{Diff}^1(M)$ and $\epsilon > 0$, for μ -a.e. $x \in M$ there exists $g \in \mathcal{U}(f)$ and a g -periodic point p such that*

$$\text{dist}(g^j(p), f^j(x)) < \epsilon \quad j = 0, 1, \dots, n(p)$$

where $n(p)$ is the period of p .

Moreover, there exists a residual set $\mathcal{D} \subset \text{Diff}^1(M)$ such that if $f \in \mathcal{D}$ and μ is an ergodic probability measure invariant under f there exists a sequence of f -periodic points p_n with periods $n(p_n)$ such that the measures $\mu_{p_n} = \frac{1}{n(p_n)} \sum_{j=0}^{n(p_n)-1} \delta_{f^j(p_n)}$ converges to μ in the weak topology.

The second part appeared in [M6]. For a surprisingly simple and elegant proof of the above theorem see [C1]. We can see now an important result regarding dominated splitting in [M5].

Theorem 3.2. *Let $f \in \text{Diff}^1(M)$ and let Λ be a compact invariant set having a Dominated Splitting $T_\Lambda M = E \oplus F$ of index i . Then, if this decomposition is not hyperbolic then for any neighborhood U of Λ and every neighborhood $\mathcal{U} \subset \text{Diff}^1(M)$ of f there exist $g \in \mathcal{U}$ and a hyperbolic g -periodic point p in U with index different from i .*

Sketch of proof: If the Dominated Splitting is not hyperbolic, then F is not uniformly expanded or E is not uniformly contracted. Assume that E is not uniformly contracted. Then there exists a point $x \in \Lambda$ such that $\prod_{j=0}^n \|Df_{/E_{f^j(x)}}\| \geq \frac{1}{2}$ for any $n \geq 0$. Consider the sequence of probability measures $\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$. Let μ an accumulation point of these measures. Then $\int \log(\|Df_{/E_x}\|) d\mu \geq 0$. By the ergodic decomposition theorem, there exists an ergodic measure ν such that $\int \log(\|Df_{/E_x}\|) d\nu \geq 0$. Let z be ν generic point. Now, we may assume that U and \mathcal{U} are sufficiently small so that any invariant set of $g \in \mathcal{U}$ in U has Dominated Splitting of index i . Let $\epsilon_n \rightarrow 0$. By the Ergodic Closing Lemma there exist $g_n \in \mathcal{U}$ and p_n a periodic point of g_n such that $\text{dist}(f^j(x), g_n^j(p_n)) < \epsilon_n, j = 0, \dots, n(p_n)$. If ϵ_n is small then the g_n -orbit of p_n is in U . Then,

$$(5) \quad \prod_{j=0}^{n(p_n)-1} \|Dg_n_{/E_{g_n^j(p_n)}}\| \geq 1 - \gamma_n$$

for some $\gamma_n \rightarrow 0$ (that it is chosen in advance). By the domination ones get

$$\prod_{j=0}^{n(p_n)-1} \|Dg_n^{-1}_{/F_{g_n^j(p_n)}}\| \leq \lambda^{n(p_n)}$$

for some $\lambda < 1$.

Now, (5) implies that for large n we have that $Dg_n^{n(p_n)}_{/E_{p_n}}$ has an eigenvalue close to one or can be perturbed to have an eigenvalue (close to) one (see [M5]). Using Franks' Lemma ([F]) one can thus obtain $\tilde{g}_n \in \mathcal{U}$ such that p_n is \tilde{g}_n periodic and the index of p_n is less than i . \square

The above result says, roughly speaking, that if a Dominated Splitting of index i over a set Λ is not hyperbolic it is due to the presence of hyperbolic periodic points of index $\neq i$. A major problem is whether these periodic points of different indices are attached to Λ . Let us be more precise: assume that Λ is the homoclinic class of a hyperbolic periodic point p of index i , and so, it has a natural continuation (for g close to f consider $\Lambda(g)$ to be the homoclinic class of the continuation p_g of p). If the Dominated Splitting is not hyperbolic, then we can perturb f so that $\Lambda(g)$ has a periodic point of different index? It is possible to have counter examples of this (see Remark 5.1) and the right question should be: if the Dominated Splitting is not hyperbolic then we can perturb f so that either $\Lambda(g)$ is hyperbolic or contains a point of different index. Another way to ask the same is assuming some generic conditions on f :

Problem 1. *Let $f \in \text{Diff}^r(M)$, $(r \geq 1)$ and assume that f is C^r generic. Let p be a hyperbolic periodic point of index i and assume that the homoclinic class $H(p)$ has a Dominated Splitting of index i . If this Dominated Splitting is not hyperbolic, is it true that $H(p)$ contains a periodic point of index $\neq i$?*

This problem has been attacked in various different situations (always in the C^1 topology), and with sophisticated techniques (see for instance [C1], [C2], [CP]). A possible approach would be to show that the homoclinic class $H(p)$ has “weak periodic points of index i ” (see Problem 2). Let’s see some results in this direction. Some previous results are needed. The first one is Pliss’ Lemma, a fundamental tool to play with dominated splitting¹.

Theorem 3.3 (Pliss’ Lemma [Pl2]). *Let $f : M \rightarrow M$ be a diffeomorphism. Then, given $0 < \gamma_1 < \gamma_2 < 1$ there exists a positive integer $N = N(\gamma_1, \gamma_2) > 0$ and $d > 0$ such that if for some $x \in M$ and some subspace $E_0 \subset T_x M$ and denoting $E_j = Df^j E_0$ we have that $\prod_{j=0}^N \|Df_{/E_j}\| \leq \gamma_1^N$ then there exist $0 \leq n_0 < n_1 < \dots < n_k < N$ with $k > dN$ such that*

$$\prod_{j=0}^{n-1} \|Df_{/E_{n_i+j}}\| \leq \gamma_2^n \quad \forall \quad 1 \leq n \leq N - n_i, \quad i = 0, \dots, k.$$

The next result is a version of the Anosov Closing Lemma. Let Λ be a compact invariant set having Dominated Splitting $T_\Lambda M = E \oplus F$. Let $0 < \eta < 1$. For $x \in \Lambda$ and a positive integer n we say that $(x, f^n(x))$ is η -string provided

$$\prod_{j=0}^{k-1} \|Df_{/E_{f^j(x)}}\| \leq \eta^j, \quad k = 1, \dots, n \quad \text{and} \quad \prod_{j=0}^{k-1} \|Df_{/F_{f^{n-j}(x)}}^{-1}\| \leq \eta^j, \quad k = 1, \dots, n$$

Theorem 3.4. *Let $f : M \rightarrow M$ be a diffeomorphism and let Λ be a compact invariant set having Dominated Splitting $T_\Lambda M = E \oplus F$ and let $0 < \eta < 1$ be given. Then, given $\epsilon > 0$ there exists $\delta > 0$ such that if (x_i, f^{n_i}) are η -strings, $i = 1, \dots, k$ such that $\text{dist}(f^{n_i}(x_i), x_{i+1}) < \delta$, for $i = 1, \dots, k-1$ and $\text{dist}(f^{n_k}(x_k), x_1) < \delta$ then there exists a periodic point p , with $f^n(p) = p$ where $n = n_1 + \dots + n_k$ and*

$$\text{dist}(f^{n_1+\dots+n_{j-1}+\ell}(p), f^\ell(x_j)) < \epsilon \quad 1 \leq j \leq k, \quad 0 \leq \ell \leq n_j.$$

When $k = 1$, an explicit proof was given by Liao [L2]. A proof of the generalized version was done by Gan [Ga1]. The following theorem is a (simplified) version of a result proved by Mañé [M1] as an important step for the stability conjecture (Theorem II.1). See also [BGY]. The proof we present here is different from the one of Mañé or [BGY], although it is based on the same ideas.

Theorem 3.5. *Let p be a hyperbolic periodic point. Assume that the homoclinic class $H(p)$ of p has a dominated splitting $T_{H(p)} M = E^s \oplus F$ of the same index as the index of p and where E^s is uniformly contracted. Then, either the homoclinic class is hyperbolic or there are periodic points in the class with arbitrarily weak Lyapunov exponent along F , or more precisely, given $0 < \gamma < 1$ there exists a periodic $q \in H(p)$ such that*

$$\gamma^{n(q)} < \prod_{j=0}^{n(q)-1} \|Df_{/F_{f^{-j}(q)}}^{-1}\| < 1$$

where $n(q)$ is the period of q .

¹Another useful tool is Liao’s Selection Lemma [L3] but we won’t state it since we won’t use it here.

Sketch of proof. Since p is hyperbolic and $\text{index}(p) = \dim E^s$ we have (perhaps changing the riemannian metric) that $\|Df_{/F_{f^j(p)}}^{-1}\| < \lambda < 1$ for some $0 < \lambda < 1$. (We also may assume that $\|Df_{/E_x}\| < \lambda$ for any $x \in H(p)$ since E^s is uniformly contracted). From this, it is not difficult to find a dense set \mathcal{D} in $H(p)$ consisting of periodic orbits such that for any $q \in \mathcal{D}$ we have $\prod_{j=0}^{n(q)-1} \|Df_{/F_{f^{-j}(q)}}^{-1}\| < \lambda_1^{n(q)}$ for some $\lambda < \lambda_1 < 1$. Now, fix any γ such that $\lambda_1 < \gamma < 1$ and assume the homoclinic class is not hyperbolic. Choose $\gamma < \gamma_1 < \gamma_2 < 1$. From Pliss' Lemma one gets that for every $q \in \mathcal{D}$ there are points q_i in the orbit of q such that

$$(6) \quad \prod_{j=0}^{n-1} \|Df_{/F_{f^{-j}(q_i)}}^{-1}\| \leq \gamma_2^n \text{ for any } n \geq 1.$$

For each $q \in \mathcal{D}$ consider all the points $q_1, q_2, \dots, q_{m_q} = q_1$ that satisfies (6) and ordered following the orbit of q . Notice that for any q_i, q_j both (q_i, q_j) and (q_j, q_i) are γ_2 -strings. Now lets look at the orbit distance between q_i and q_{i+1} , that is m_i , where $f^{m_i}(q_i) = q_{i+1}$. If m_i is uniformly bounded for any i and for every $q \in \mathcal{D}$ we conclude that $H(p)$ is uniformly hyperbolic. Thus, we may find a sequence $q(n)$ in \mathcal{D} and points $q_{i_n}(n), q_{i_n+1}(n)$ (in the orbit of $q(n)$) whose orbit distance goes to infinity. And so, for n large enough, we get from Pliss Lemma that $\prod_{j=0}^{m_i-1} \|Df_{/F_{f^{-j}(q_{i_n+1}(n))}}^{-1}\| > \gamma_1^{m_i}$ (otherwise we have a point satisfying (6) between $q_{i_n}(n)$ and $q_{i_n+1}(n)$).

Now we may assume that $q_{i_n}(n)$ and $q_{i_n+1}(n)$ converge to points x and y . Let $\epsilon > 0$ be very small and let δ form the above closing lemma. Fix some n_0 such that $q_{i_{n_0}}(n_0)$ and $q_{i_{n_0}+1}(n_0)$ are at distance less than $\delta/2$ of x and y . And then, choose n_1 much larger than n_0 so that $q_{i_{n_1}}(n_1)$ and $q_{i_{n_1}+1}(n_1)$ are at distance less than $\delta/2$ of x and y . Notice that the γ_2 strings $(q_{i_{n_0}+1}(n_0), q_{i_{n_0}}(n_0))$ and $(q_{i_{n_1}}(n_1), q_{i_{n_1}+1}(n_1))$ are as in the Theorem 3.4 with $k = 2$. Choose $0 < c < 1$ such that $\gamma < (1-c)\gamma_1 < (1+c)\gamma_2 < 1$ and set $N = m_{i_{n_0}} + m_{i_{n_1}}$ and let z be the periodic point that shadows these two strings by Theorem 3.4, $f^N(z) = z$.

Let $C = \sup_{x \in H(p)} \|Df_{/F_x}^{-1}\|^{-1}$. If n_1 was chosen large enough then we have

$$C^{m_{i_{n_0}}} ((1-c)\gamma_1)^{m_{i_{n_1}}} > \gamma^N.$$

Thus, if ϵ was small enough we conclude that $\gamma^N < \prod_{j=0}^{N-1} \|Df_{/F_{f^{-j}(z)}}^{-1}\|$. On the other hand $\prod_{j=0}^{n-1} \|Df_{/F_{f^{-j}(z)}}^{-1}\| \leq ((1+c)\gamma_2)^n$ for $1 \leq n \leq N$. This implies that z has a uniform unstable manifold and therefore, and if ϵ and δ were chosen sufficiently small, we have a heteroclinic intersection between z and $q_{i_{n_0}+1}(n_0)$ and so z belongs to the homoclinic class $H(p)$. Since γ was arbitrary we deduce the existence of periodic points with arbitrarily weak Lyapunov exponent along F .

□

Problem 2. Let p be a hyperbolic periodic point and let $H(p)$ be its homoclinic class. Assume that $H(p)$ has Dominated Splitting $T_{H(p)}M = E \oplus F$ of index equal to the index of p . Is it true that if neither E is uniformly contracting nor F is uniformly expanding, there are periodic points in the class having a weak Lyapunov exponent along E and periodic points having weak Lyapunov exponent along F ? (See Problem 1.8 of [CSY])

A partial answer to the above Problem can be given from the results in [CSY]

Theorem 3.6. Let $f \in \text{Diff}^1(M)$ be a generic diffeomorphism. Let p be a hyperbolic periodic point and assume that the homoclinic class $H(p)$ has dominated splitting $T_{H(p)} = E^s \oplus E^c \oplus F$ where E^s is uniformly contracted, E^c is one dimensional and not uniformly contracted, and $\dim(E^s \oplus E^c)$ is equal to the index of p . Then, for any $\delta > 0$ there exists periodic points in $H(p)$ whose Lyapunov exponent along E^c belongs to $(-\delta, 0)$.

We can refine Problems 1 and 2 and ask the following:

Problem 3. *Let $H(p)$ be a homoclinic class of a hyperbolic periodic point having a Dominated Splitting $T_{H(p)} = E^s \oplus E_1^c \oplus \dots \oplus E_k^c \oplus E^u$ where E^s is uniformly contracted and E^u is uniformly expanded and E_i^c are one dimensional and neither E_1^c is uniformly contracted nor E_k^c is uniformly expanded. Do there exist periodic points q_1, q_2 in $H(p)$ such that the index of q_1 is $\dim E^s$ and the index of q_2 is $\dim E^s + k$? To avoid some counterexamples one should assume that f satisfies some C^r generic condition (see the end of Section 5).*

We remark that from [CSY] the indices $\dim E^s + 1$ and $\dim E^s + k - 1$ are realized (and by [ABCDW] the indices between both are realized as well). The above problem admits a more general formulation (and more difficult?) when the class $H(p)$ admits a Dominated Splitting $T_{H(p)} = E^s \oplus E^c \oplus E^u$ where E^s and E^u are maximal (i.e., $H(p)$ does not admit a dominated splitting $\tilde{E}^s \oplus \tilde{E}^c \oplus \tilde{E}^u$ with $E^s \subset \tilde{E}^s$ and $E^u \subset \tilde{E}^u$). A positive answer to the above problem will represent a fundamental step towards the Palis' conjecture or Bonatti's conjecture on finiteness of chain-recurrent classes far from tangencies (see [B]).

4. INTERMEDIATE ONE DIMENSIONAL SUBBUNDLE

Let $f : M \rightarrow M$ be a diffeomorphism and let Λ be a compact invariant set having a dominated splitting $T_\Lambda = E \oplus E^c \oplus F$ where E^c is one dimensional. Can we say something about the dynamics of Λ ? Even in the case where $E = E^s$ is uniformly contracting and $F = E^u$ is uniformly expanding this is a difficult question.

Problem 4. *Assume that $f \in \text{Diff}^r(M)$ is C^r generic and partially hyperbolic $TM = E^s \oplus E^c \oplus E^u$ with E^c one dimensional (and f is not Anosov). Is it true that the chain recurrent set $\mathcal{R}(f)$ has finitely many chain-recurrent classes? What if $TM = E^s \oplus E_1^c \oplus \dots \oplus E_k^c \oplus E^u$ with E_i^c one dimensional?*

Related problems to the above are for instance: Let f be time one map of an Anosov flow, is it true that any C^r generic perturbation has finitely many chain-recurrent class? Is any C^r perturbation transitive?

There are two main reasons to consider dominated splitting with one-dimensional subbundle. On one hand this is the situation when we are far from homoclinic tangencies (see [PS2], [We1], [C2]). On the other one, although the action of Df along E^c might be neutral, since it is one dimensional, one could expect to have a description on the dynamics in the E^c direction. This is the content of a basic tool in the area known as Crovisier's Center Models and developed in [C2] and [C3]. This tool was used to prove the major results on the Palis' conjecture ([C3], [C2] and [CP]) and the Bonatti's conjecture ([B]) on finiteness of chain-recurrent classes far from tangencies as well (see [CSY]).

Let us briefly explain this tool (we recommend to look at [C1] for this and many applications as well). By the theory in [HPS] one knows that there are a family of one dimensional manifolds $\mathcal{W}^c(x)$ for $x \in \Lambda$ (center plaques) such that $T_x \mathcal{W}^c(x) = E_x^c$ and are locally invariant, that is, $f(\mathcal{W}^c(x))$ contains a neighborhood of $f(x)$ within $\mathcal{W}^c(f(x))$. This allows to lift the dynamics to $T_\Lambda E^c$.

Definition 4.1. A center model $(\hat{\Lambda}, \hat{f})$ associated to a compact invariant set Λ and f with $T_\Lambda = E \oplus E^c \oplus F$ where $\dim E^c = 1$ is a compact set $\hat{\Lambda}$, a continuous map $\hat{f} : \hat{\Lambda} \times [0, +\infty)$ and a map $\pi : \hat{\Lambda} \times [0, +\infty) \rightarrow M$ such that:

- $\hat{f}(\hat{\Lambda} \times \{0\}) = \hat{\Lambda} \times \{0\}$.
- \hat{f} is a local homeomorphism of a neighborhood of $\hat{\Lambda} \times \{0\}$.

- \hat{f} can be written as $\hat{f}(x, t) = (\hat{f}_1(x), \hat{f}_2(x, t))$.
- $\pi(\hat{\Lambda} \times \{0\}) = \Lambda$. item $\pi \circ \hat{f} = f \circ \pi$.
- The maps $t \rightarrow \pi(\hat{x}, t)$ forms a family of C^1 embedding of $[0, +\infty)$ in M .
- The curve $\pi(\hat{x} \times [0, +\infty))$ is tangent to E^c at $\pi(\hat{x}, 0)$.

One can consider two cases: the orientable case where there exists an orientation of E^c invariant by Df and where there is not.

Theorem 4.1 ([C2],[C3]). *Let Λ be a compact, invariant and chain-transitive set with Dominated Splitting $T_\Lambda = E \oplus E^c \oplus F$ where $\dim E^c = 1$. Then there exists a center model $(\hat{\Lambda}, \hat{f})$ associated to (Λ, f) where $\hat{\Lambda}$ is chain-transitive. In the orientable case, $\pi_{/\hat{\Lambda} \times 0}$ is a homeomorphism and $\pi(\{\hat{x}\} \times [0, +\infty))$ is compatible with the orientation on E^c , and in the non-orientable case $\hat{\Lambda}$ coincides with $T_\Lambda^1 E^c$. Moreover, one of the following cases holds:*

- Type(R): (Chain-recurrent) For every $\epsilon > 0$ there exists a point $x \in \Lambda$ and an arc $\gamma, x \in \gamma \subset \mathcal{W}^c(x)$ such that the length of $f^n(\gamma)$ is bounded by ϵ and one can go from γ to Λ and viceversa with arbitrarily small chains contained in a neighborhood of Λ .
- Type(N): (Neutral) There exist a base of attracting open neighborhoods U of the zero section (i.e., $\hat{f}(\bar{U}) \subset U$) and a base of repelling open neighborhoods.
- type(H): (Hyperbolic) There exists a base of attracting (respect. repelling) open neighborhoods of the zero section and \mathcal{W}^c is contained in the chain stable (respect. unstable) set of Λ .
- Type(P): (Parabolic) In this case we are in the orientable case and on each “side” we have type (N) or (H). Thus, three subcases appear: P_{SU}, P_{NS}, P_{NU} depending on which cases appear on $E^{c,+}$ and $E^{c,-}$.

As we said before, using the central models one can get many interesting result. Just to give a rough idea let's see the following:

Theorem 4.2 ([C2],[C3]). *Let $f \in \text{Diff}^1$ be a C^1 generic diffeomorphism and let Λ be a compact invariant chain-transitive set with Dominated Splitting $T_\Lambda = E^s \oplus E^c \oplus E^u$ where E^s is uniformly contracted, E^u is uniformly expanded and $\dim E^c = 1$ and of Type(R). Then, Λ is contained in the chain-recurrent class of a periodic point.*

Sketch of Proof: Every point whose orbit remains in a neighborhood of Λ has a uniform (strong) stable \mathcal{W}^s and unstable \mathcal{W}^u manifolds (tangent to E^s and E^u). Let γ be a chain recurrent segment. Then $\cup_{c \in \gamma} \mathcal{W}^s(x)$ and $\cup_{c \in \gamma} \mathcal{W}^u(x)$ defines (topological) manifolds of dimension $\dim E^s + 1$ and $\dim E^u + 1$ and contained respectively in the chain stable and chain unstable set of Λ . Since f is generic (see [BC] and [C4]) one find a periodic orbit \mathcal{O} whose orbit remains in a neighborhood of Λ and there is $p \in \mathcal{O}$ close to the middle point of γ . Thus, the strong stable and unstable manifolds at p intersects the chain unstable and chain stable set of Λ . \square

The following result is a consequence of Theorem 1.2 of [CSY]:

Theorem 4.3. *Let f be a C^1 generic diffeomorphism and let Λ be a compact invariant chain transitive set having a Dominated Splitting $T_\Lambda M = E^s \oplus E_1^c \oplus \dots \oplus E_k^c \oplus E^u$ where E^s is uniformly contracted, E^u is uniformly expanded and $\dim E_i^c = 1, i = 1, \dots, k$. And neither E_1^c is uniformly contracted nor E_k^c is uniformly expanded. Then one of the following holds:*

- $k = 1$ and the Lyapunov exponent along E_1^c of every ergodic measure supported in Λ is zero.
- Λ is contained in the chain-recurrent class of a periodic point.

The idea is the following: if all Lyapunov exponents along E_1^c of every ergodic measure are zero then we are in the first case. Otherwise there exists an ergodic measure μ whose Lyapunov exponent is non-zero and Theorem 1.2 of [CSY] applies and we are in the second case.

Furthermore, it follows from the same theorem that, in the second case, Λ is contained in the local homoclinic class of a hyperbolic periodic orbit \mathcal{O} in a neighborhood U of Λ (that is, the closure of the set of transversal intersections of the stable and unstable manifolds of \mathcal{O} whose orbits are contained in U) and so with the same structure $E^s \oplus E_1^c \oplus \dots \oplus E_k^c \oplus E^u$.

Problem 5. *Let f be a C^r generic diffeomorphism. Does there exist an aperiodic chain recurrent class \mathcal{C} (i.e. a chain recurrent class without periodic points) partially hyperbolic $T_{\mathcal{C}}M = E^s \oplus E^c \oplus E^u$ with $\dim E^c = 1$?*

A negative answer to this question will represent a definite step towards the Palis's conjecture or the Bonatti's conjecture (see [B], [CP], [CSY]).

5. EXTREMAL ONE DIMENSIONAL SUBBUNDLE

In this section we study compact invariant sets Λ having dominated splitting $T_{\Lambda}M = E \oplus F$ where E or F is one dimensional. We call this kind of splitting *codimension one dominated splitting*.

Lets begin with the simplest case: when the ambient manifold M is compact surface (i.e. a bidimensional compact riemannian manifold). If Λ is a compact set having dominated splitting $T_{\Lambda}M = E \oplus F$ then both E and F are (extremal and) one dimensional. How such a set can fail to be hyperbolic? There are two trivial counterexamples: either Λ contains a non hyperbolic periodic point or Λ contains a periodic simple closed curve normally hyperbolic (and hence attracting or repelling) such that at the period the dynamics has irrational rotation number. The next result says that these are the only obstructions provided the diffeomorphism is at least of class C^2 .

Theorem 5.1 ([PS2]). *Let $f \in \text{Diff}^2(M)$ where M is bidimensional. Let Λ be a compact invariant set having a Dominated Splitting $T_{\Lambda}M = E \oplus F$. Assume that any periodic point of f in Λ is hyperbolic and that Λ does not contain a periodic simple closed curve supporting an irrational rotation. Then Λ is hyperbolic.*

This result implies that if all periodic points hyperbolic and no normally hyperbolic curve supporting an irrational rotation exists (which is a C^r generic condition for $r \geq 1$) then a set having dominated splitting for $f \in \text{Diff}^r, r \geq 2$ is hyperbolic. And also implies that there exists a residual set $\mathcal{D} \subset \text{Diff}^1(M)$ such that if $f \in \mathcal{D}$ and Λ is a compact invariant set having dominated splitting is hyperbolic. To prove this consider $\{U_n\}$ a countable basis of the topology of M . Consider the family $\{\mathcal{V}_n\}$ of finite collection of elements U_n 's (which is countable). Let \mathcal{A}_n the interior of the set

$$\{f \in \text{Diff}^1(M) : \text{if } K \subset \mathcal{V}_n \text{ is compact, invariant having DS is hyperbolic}\}$$

where DS stands for Dominated Splitting. Let \mathcal{B}_n be the complement of the closure of \mathcal{A}_n . Let $\mathcal{D}_n = \mathcal{A}_n \cup \mathcal{B}_n$ and $\mathcal{D} = \bigcap_n \mathcal{D}_n$. Let $f \in \mathcal{D}$ and let K a compact invariant set having Dominated Splitting. Let V be a compact neighborhood of K and let \mathcal{U} be a neighborhood of f such that for any $g \in \mathcal{U}$ the maximal invariant set of g in V has Dominated Splitting. Now, there exists \mathcal{V}_n such that $K \subset \mathcal{V}_n \subset V$. Taking a sequence g_n of C^2 generic diffeomorphism converging to f we see that the maximal invariant set of g_n in V is hyperbolic and so f can not be in \mathcal{B}_n . Therefore, $f \in \mathcal{A}_n$ and so K is hyperbolic.

So far there is no proof of this fact just using C^1 techniques (i.e. without approximating by a C^2 diffeomorphism and using Theorem 5.1).

The proof of Theorem 5.1 is very technical. It is an extension of a result on non-critical one dimensional dynamics of Mañé ([M7]). A very rough and general idea about the proof is: since both E and F are one dimensional extremal bundles, there exist locally invariant manifolds $\mathcal{W}_{loc}^{cs}(x)$ and $\mathcal{W}_{loc}^{cu}(x)$ tangent to E and F respectively having dynamics properties: if $y \in \mathcal{W}_{loc}^{cs}(x)$ then $\text{dist}(f^n(y), f^n(x)) \rightarrow_{n \rightarrow \infty} 0$ and similarly for $\mathcal{W}_{loc}^{cu}(x)$ in the past (in terms of Crovisier's Center Models, they are respectively type(H)-attractive and type(H)-repelling). Then one prove that $\sum_{n \geq 0} |f^n(\mathcal{W}_{loc}^{cs}(x))| < \infty$ where $|f^n(\mathcal{W}_{loc}^{cs}(x))|$ denotes the length (the same for $\mathcal{W}_{loc}^{cu}(x)$ in the past). Finally, the above implies that $\|Df^n_{/E_x}\| \rightarrow_{n \rightarrow \infty} 0$ and $\|Df^{-n}_{/F_x}\| \rightarrow_{n \rightarrow \infty} 0$. For all these facts, the C^2 assumption is crucial (in order to control distortion). Indeed, if f is just C^1 the above theorem is false: at the end of this section we give a counterexample.

Theorem 5.1 gives that dominated splitting on two dimensions of a C^2 diffeomorphism imposes certain constraints. Thus, we may ask if we can fully describe the dynamics. The answer is *yes* (at least to some extent).

Theorem 5.2 ([PS3]). *Let $f \in \text{Diff}^2(M^2)$ and assume that the Limit Set $L(f)$ has a dominated splitting. Then $L(f)$ can be decomposed into $L(f) = \mathcal{I} \cup \tilde{L}(f) \cup \mathcal{R}$ such that*

- (1) \mathcal{I} is a set of periodic points with bounded periods and contained in a disjoint union of finitely many normally hyperbolic periodic arcs or simple closed curves.
- (2) \mathcal{R} is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.
- (3) $\tilde{L}(f)$ can be decomposed into a disjoint union of finitely many compact invariant and transitive sets. The periodic points are dense in $\tilde{L}(f)$ and contains at most finitely many non-hyperbolic periodic points. The (basic) sets above are the union of finitely many (nontrivial) homoclinic classes. Furthermore $f/\tilde{L}(f)$ is expansive.

A fundamental step to prove the theorem above is the following rather surprising result:

Theorem 5.3 ([PS3]). *Let $f : M \rightarrow M$ be a C^2 -diffeomorphism of a two dimensional compact riemannian manifold M and let Λ be a compact invariant set having dominated splitting. Then, there exists an integer $N_1 > 0$ such that any periodic point $p \in \Lambda$ whose period is greater than N_1 , is a hyperbolic periodic point of saddle type.*

Now, let turn to $\dim M \geq 3$ and let Λ a compact invariant set having codimension one dominated splitting, say $T_\Lambda M = E \oplus F$ where F is one-dimensional. A natural extension of Theorem 5.1 should be: similar conditions as in Theorem 5.1 imply that F is uniformly expanding?. Some partial results were given in [PS4] (when E is uniformly contracted) and in [CP] with some condition on the topology of Λ . Nevertheless, it turns out to be true:

Theorem 5.4 ([CPS]). *Let $f : M \rightarrow M$ be a C^2 diffeomorphism and let Λ be a compact invariant set having a dominated splitting $T_\Lambda M = E \oplus F$ with $\dim F = 1$. Assume that for any periodic points in Λ the Lyapunov exponent F is positive and that Λ does not contain a periodic simple closed curve tangent to F and supporting an irrational rotation. Then F is uniformly expanded.*

We finish this section giving a C^1 counterexample for Theorem 5.1:

Theorem 5.5. *There exists a C^1 diffeomorphism $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ having a Dominated Splitting $T\mathbb{T}^2 = E \oplus E^u$ such that E^u is uniformly expanded and:*

- Any periodic point of g is hyperbolic of saddle type

- g is in the C^1 closure of the set of Anosov diffeomorphisms.
- g is conjugated to an Anosov diffeomorphism.
- E is not uniformly contracted.

For the (sketch of the) proof of this theorem we need some auxiliary lemmas. The first one is straightforward.

Lemma 5.1. *Given, $\sigma > 1$ there exist $\eta > 0, \beta > 0$ and $\sigma_1 > 1$ such that given h a local diffeomorphism around 0 in \mathbb{R}^2 , $h(0) = 0$ and assume that h can be written as $h(x, y) = (u(x, y), v(y))$ and such that $\|u_x\| \leq e^\beta$, $\|u_y\| < \eta$ and $\|v_y\| > \sigma$ then:*

- Denoting $\mathcal{C}_a^u = \{(u, v) \in \mathbb{R}^2 : \|u\| \leq a\|v\|\}$ we have $Dh_{(x,y)}\mathcal{C}_1^u \subset \mathcal{C}_\rho^u$ where $\rho = \frac{e^\beta + \eta}{\sigma} < 1$.
- If $w \in \mathcal{C}_1^u$ then $\|Dh_{(x,y)}w\| > \sigma_1$.

The next lemma will be the key to the induction argument we are going to do:

Lemma 5.2. *Let $h(x, y) = (u(x, y), v(y))$ be as the above Lemma. Let $\delta > 0, \alpha > 0, \gamma > 0, \beta > 0$ and $\lambda > -\beta$ be given with $e^\beta < 2$ and assume that $0 < u_x \leq e^{-\lambda}$ then there exists a local diffeomorphism g which coincides with h outside a ball of radius δ at the origin and such that g can be written as $g(x, y) = (\tilde{u}(x, y), v(y))$ and such that:*

- $\|\tilde{u}_x\| \leq e^{\gamma-\lambda}$ and $\|\tilde{u}_y\| < \eta$.
- $\text{dist}_{C^0}(h, g) < \alpha$.
- $\text{dist}_{C^1}(h, g) < 2(e^\gamma - 1)$.
- $Dg_{(0,0)}(1, 0) = (\tilde{u}_x(0, 0), 0) = e^\gamma(u_x(0, 0), 0) = e^\gamma Dh_{0,0}(1, 0)$.

Proof. Let $Z : \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ bump function such that $Z(0) = 1$, $\text{supp}[Z] \subset (-\frac{\delta_1}{2}, \frac{\delta_1}{2})$ (where $\text{supp}[Z]$ is the support of Z) and $|Z'(t)| < \frac{4}{\delta_1}$ where $\delta_1 = \delta/2$.

We need the following auxiliary lemma (See Lemma 2.0.1 of [PaS])

Sublemma: *For all $k > 0$ arbitrarily small and $\gamma > 0$ there exist a function $\beta_k : [0, +\infty) \rightarrow \mathbb{R}$ such that:*

- (1) β_k is C^∞ , non-increasing and such that $-k \leq \beta'_k(t)t \leq 0$.
- (2) β_k is supported in $[0, k]$, i.e. $\text{supp}[\beta_k] \subset [0, k]$.
- (3) $\beta(0) = e^\gamma - 1 > 0$.

Define

$$g(x, y) = h(x, y) + (Z(y)\beta(x^2)u_x(0, 0)x, 0).$$

Notice that $\tilde{u}(x, y) = u(x, y) + Z(y)\beta(x^2)u_x(0, 0)x$. Let's see that the conditions are fulfilled if k in the above sublemma was chosen small enough. It is immediately that g and h coincides outside a ball of radius δ at the origin.

Notice that $\|g(x, y) - h(x, y)\| \leq \beta(0)e^\beta k < \alpha$ if k is small enough. Now, $Dg = Dh + A$ with $A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}$ and

$$a_1 = Z(y)[\beta(x^2)u_x(0, 0) + 2\beta'(x^2)x^2u_x(0, 0)] \quad \text{and} \quad a_2 = Z'(y)\beta(x^2)u_x(0, 0)x.$$

Then, $\|a_1\| \leq \beta(0)e^{-\lambda} + 2ke^{-\lambda} < 2\beta(0)$ if k is small. Observe that $\beta(0) = e^\gamma - 1$. On the other hand $\|a_2\| \leq \frac{4}{\delta_1}\beta(0)e^{-\lambda}k < \beta(0)$ again if k is small enough.

Moreover $\tilde{u}_x(0, 0) = u_x(0, 0) + \beta(0)u_x(0, 0) = e^\gamma u_x(0, 0)$. Finally, since $u_x > 0$ and $-k \leq \beta'(t)t \leq 0$ we have that $\tilde{u}_x > 0$ if k is small and

$$\|\tilde{u}_x\| \leq \|u_x\| + \beta(0)\|u_x(0, 0)\| \leq e^{-\lambda}(1 + \beta(0)) = e^{\gamma-\lambda}$$

□

Let's continue with the proof of Theorem 5.5. The idea is to begin with a linear Anosov and then perform a sequence of perturbations (along periodic points) that converge in the C^1 topology to the desired diffeomorphism.

Start with the linear Anosov diffeomorphism $f_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Let $e^{-\lambda}$ and e^μ be the eigenvalues of A where λ and μ are positive and let E^s, E^u be the associated subspaces. Everything we do in local coordinates will be referred to the decomposition $E^s \oplus E^u$ (both in the tangent space as in \mathbb{T}^2). Choose $1 < \sigma < e^\mu$ and then set $\beta > 0, \eta > 0$ and σ_1 from the first lemma (and ρ as well).

Choose a sequence $\epsilon_n, n \geq 1$ such that $\sum_{n \geq 1} \epsilon_n < \beta$. Now, set a sequence of positive numbers $\lambda_n, n \geq 0$ as follows: $\lambda_0 = \lambda$ and for $n \geq 1$:

$$\frac{\lambda_{n-1}}{2} < \lambda_n < \frac{\lambda_{n-1}}{2} + \epsilon_n.$$

Let $\tilde{\lambda}_n, n \geq 0$ be a sequence such that

$$\tilde{\lambda}_0 = \lambda_0 \quad \text{and} \quad \frac{\lambda_{n-1}}{2} < \lambda_n < \tilde{\lambda}_n < \frac{\lambda_{n-1}}{2} + \epsilon_n \quad \text{for } n \geq 1.$$

Set $\gamma_n = \tilde{\lambda}_n - \lambda_{n+1}, n \geq 0$ and observe that $\sum_{n \geq 0} \gamma_n < \lambda_0 + \sum \epsilon_n < \infty$ and that $\sum(\tilde{\lambda}_n - \lambda_n) < \sum \epsilon_n < \beta$.

Now, let's begin our induction process. Let $p = p_0$ the fixed point of f_0 . Applying Lemma 5.2 around the fixed point p_0 with $\lambda = \lambda_0$ and $\gamma = \gamma_0$ (δ and α do not matter too much here), we get a diffeomorphism $f_1 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ where the stable foliation (tangent to E^s) is kept invariant, p_0 is fixed by f_1 , and

- The Lyapunov exponent of f_1 at p_0 is $L^s(p_0, f_1) = -\lambda_0 + \gamma_0 = -\lambda_1$. That is $\|Df_1|_{E^s}\| = e^{-\lambda_1}$.
- $\text{dist}_{C^1}(f_0, f_1) \leq 2(e^{\gamma_0} - 1)$.
- $\|Df_1|_{E^s_x}\| \leq e^{\gamma_0 - \lambda_0} \leq e^{\lambda_1} < 1 < e^\beta$. for all $x \in \mathbb{T}^2$.
- f_0 and f_1 coincides outside a δ -neighborhood of p_0 .
- f_1 has a dominated splitting (by Lemma 5.1) with expanding direction in the cone \mathcal{C}_ρ^u
- f_1 is Anosov.

The true induction process starts here: pick a periodic point p_1 of f_1 with large period n_1 that spends much of its time near p_0 so that its (stable) Lyapunov exponent is $L^s(p_1) = \frac{1}{n_1} \log \|Df_1|_{E^s_{p_1}}\| = -\tilde{\lambda}_1$ for some $\tilde{\lambda}_1$ where $\lambda_1 < \tilde{\lambda}_1 < \lambda_0/2 + \epsilon_1$.

Now, pick $\delta > 0$ such that $f_1^j(B_\delta(f^i(p_1))), j = 0, \dots, n_1 - 1$ are disjoint for any $i = 0, \dots, n - 1$ and that p_0 does not belong to $B_\delta(\mathcal{O}(p_1)) = \cup_{i=0}^{n_1-1} B_\delta(f^i(p_1))$. Let $\alpha_1 > 0$ be such that if $\text{dist}_{C^0}(f_1, g) \leq 2\alpha_1$ then $g^j(B_\delta(f^i(p_1))), j = 0, \dots, n_1 - 1$ are disjoint for any $i = 0, \dots, n - 1$.

Putting local coordinates at $f^i(p_1)$ and $f^{i+1}(p_1)$ for $i = 0, \dots, n_1 - 1$ we perform a perturbation using Lemma 5.2 with $\lambda = \lambda_1, \gamma = \gamma_1$ and we get $f_2 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that

- $f_2^j(p_1) = f_1^j(p_1), j \geq 0$.
- The stable foliation (tangent to E^s) is kept invariant.
- $L^s(p_1, f_2) = \gamma_1 + L^s(p_1, f_1) = \gamma_1 - \tilde{\lambda}_1 = -\lambda_2$.
- $\|Df_2|_{E^s_x}\| \leq e^{\gamma_1 + \gamma_0 - \lambda_0} < e^{\sum \epsilon_n} < e^\beta$.
- $\text{dist}_{C^0}(f_1, f_2) < \alpha_1$.
- $\text{dist}_{C^1}(f_1, f_2) \leq 2(e^{\gamma_1} - 1)$.

- f_2 has dominated splitting (by Lemmas 5.1 and 5.2) with expanding direction in the cone C_ρ^u
- f_2 is Anosov.

The last point deserves a bit explanation (since at some point in the process the inequality $\|Df_{n+1/E_x^s}\| \leq e^{\gamma_n + \dots + \gamma_1 + \gamma_0 - \lambda_0}$ does not ensure that $\|Df_{n+1/E_x^s}\| < 1$ although is less than e^β enough to guarantee the domination). The general argument is as follows: since f_1 is Anosov, we know that for some $m > 0$ we have $\|Df_{1/E_x^s}^m\| < 1$ for any x ; on the other hand p_1 is a periodic point of period n_1 with $L^s(p_1, f_1) = -\lambda_1$ and so, for some neighborhood U of the orbit of p_1 we have that $\|Df_{1/E_x^s}^{n_1}\| \leq e^{(-\lambda_1 + \epsilon)n_1}$ where ϵ is such that $-\lambda_2 + \epsilon < 0$: finally perform the perturbation in a tiny neighborhood of the orbit of p_1 such that any point outside U is outside the support of the perturbation in the next m iterates. The above implies that f_2 is Anosov, since if $x \in U$ then $\|Df_{2/E_x^s}^{n_1}\| \leq e^{(-\lambda_2 + \epsilon)n_1}$ and if $x \notin U$ the $\|Df_{2/E_x^s}^m\| = \|Df_{1/E_x^s}^m\| < 1$.

One last remark on how is chosen the $\alpha_n = \text{dist}_{C^0}(f_n, f_{n+1})$ (where α_n plays the role for f_n as α_1 with f_1): $\sum_{n>m} \alpha_n \leq \alpha_m$.

Therefore, inductively we have a sequence f_n of Anosov diffeomorphism on \mathbb{T}^2 such that $\text{dist}_{C^1}(f_n, f_{n+1}) \leq 2(e^{\gamma_n} - 1)$, $\text{dist}_{C^0}(f_n, f_{n+1}) \leq \alpha_n$, and f_n has a periodic point p_n whose stable Lyapunov exponent is $-\lambda_n$. The sequence f_n is also uniformly dominated (i.e same cones and estimates) and the foliation tangent to E_A^s is kept invariant. Since $\sum \gamma_n < \infty$ we have that $\{f_n\}$ is a C^1 -Cauchy sequence and thus converges to a C^1 diffeomorphism g , having a dominated splitting on \mathbb{T}^2 . The diffeomorphism g is not Anosov since g has a sequence p_n of periodic points whose (stable) lyapunov exponent converges to zero. Every periodic point of g is hyperbolic: let q be periodic of period k , then for $n \geq k$ we have that q is disjoint of the support of perturbation of f_n (since $d_{C^0}(g, f_n) \leq \alpha_n$) and so $g = f_k$ along the orbit of q and so it is hyperbolic. It is not difficult to see that g is expansive and so it is conjugated to Anosov [Le] (or see that the lift of g to \mathbb{R}^2 has infinity as expansivity constant). This concludes the proof of Theorem 5.5.

Remark 5.1. The above example does not satisfy certain C^1 generic conditions. For instance, it has no periodic attractor but by an arbitrarily small perturbation one can create them.

Notice also that by multiplying this example with a strong contraction we have a diffeomorphism on a manifold having a homoclinic class with dominated splitting which is not hyperbolic and by perturbation we can not have a periodic point in the class of different index! (recall Problem 1). However, the homoclinic class can be perturbed to be hyperbolic Besides, it also gives an example of a homoclinic class $H(p)$ with dominated splitting $T_{H(p)}M = E^s \oplus E^c \oplus E^u$ where E^c is not contracted but no periodic point in the class has index $\dim E^s$ (see Problem 3). Taking into the account the results of this section, one may ask: *does smoothness has a role to play in Problems 3 and 5?*

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